

Model-independent approach to $\eta \rightarrow \pi^+\pi^-\gamma$ and $\eta' \rightarrow \pi^+\pi^-\gamma$

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Abstract

We present a new, model-independent method to analyze radiative decays of mesons to a vector, isovector pair of pions of invariant mass square below the first significant $\pi\pi$ threshold in the vector channel. It is based on a combination of chiral perturbation theory and dispersion theory. This allows for a controlled inclusion of resonance physics without the necessity to involve vector meson dominance explicitly. As an example, the method is applied to an analysis of the reactions $\eta \rightarrow \pi^+\pi^-\gamma$ and $\eta' \rightarrow \pi^+\pi^-\gamma$.

1. Radiative decays are known to be very sensitive tools to explore decay mechanisms. Especially, when studied together with two hadrons as decay products, they enable us to adjust the invariant mass of the two-hadron system via a variation of the photon energy without interference of strong three-body final state interactions.

We here present a new, model-independent method to analyze the mentioned radiative decays. The method is based on a combination of chiral perturbation theory (ChPT) – extended from $SU(3)$ to $U(3)$ [1,2] – and a dispersive analysis. It is general and can be applied to all decays of mesons with $\pi\pi\gamma$ final states, where soft bremsstrahlung does not occur and where the pion pair is of invariant mass square below the first *significant* $\pi\pi$ threshold, which is, in

the (iso)vector case, the $\omega\pi$ threshold. In this work, however, we focus on the decay of a pseudoscalar, *i.e.* the η and η' meson, to a photon and a charged pion pair. In this case the general selection rules enforce the pion pair to be in the isovector channel. Especially for the decay of a relatively light meson, like the η , one may expect that the ideal analysis tool is the effective field theory for the standard model at low energies, ChPT, and indeed the corresponding calculation has been available to one-loop order for a long time [3]. But compared to modern data one observes a significant deviation between the theory predictions and data — the source of which is mostly the non-perturbative $\pi\pi$ final state interaction. Several efforts have been made to include final state interactions by unitarized extensions to the box-anomaly term – the latter of course determines the $\eta \rightarrow \pi\pi\gamma$ decay in the chiral limit. The tree-level calculation can be enhanced by a momentum dependent Vector Meson Dominance model [4] or more elaborate calculations in the context of Hidden Local Symmetries [5]. On top of the results at the one-loop level an Omnes-function can be applied to describe the effects of p -wave pion scattering [6,7] (see also [8]), or as done in the Chiral Unitary approach, a Bethe-Salpeter equation with coupled channels can be used to generate resonances dynamically [9]. We will later specify the details which discriminate these mechanisms from our method. The general problem with the vector meson dominance model is a priori that it is unclear what relative strength is to be put between the tree level contribution and the resonance contribution. This problem is resolved in the dispersion theory approach as we will see below.

For the transitions at hand it appears necessary to disentangle perturbative and non-perturbative effects in a controlled way. The method is therefore split in two steps. In the *first step* the spectral decay data are fitted with a function of the form (the details will be given below)

$$\frac{d\Gamma}{ds_{\pi\pi}} = |A P(s_{\pi\pi}) F_V(s_{\pi\pi})|^2 \Gamma_0(s_{\pi\pi}) , \quad (1)$$

where the normalization parameter A has the dimension of mass^{-3} and where

$$\Gamma_0(s_{\pi\pi}) = \frac{1}{3 \cdot 2^{11} \cdot \pi^3 m^3} (m^2 - s_{\pi\pi})^3 s_{\pi\pi} \sigma(s_{\pi\pi})^3$$

collects phase-space terms and the kinematics of the absolute square of the simplest gauge invariant matrix element (for point-particles). The latter is expressed through the $\pi\pi$ -two-body phase-space $\sigma(s_{\pi\pi}) = \sqrt{1 - 4m_\pi^2/s_{\pi\pi}}$ in terms of the invariant mass square $s_{\pi\pi}$ of the pion pair, while m (m_π) denotes the mass of the decaying particle (charged pion). Since the initial state is a pseudoscalar and the final state contains a photon, the partial wave of the charged pion pair is expected to be dominated by 1^{--} . If it can be confirmed by experiment that the other partial waves can be neglected, then the factorization given in Eq.(1) is exact and can be straight forwardly derived using

dispersion theory (see next section). If, however, it eventually might turn out that higher partial waves would be needed to get a precise fit of the angular data, then their contributions could still be added in a perturbative way to our model-independent expression for the p -wave decay amplitude.

The pion vector form factor $F_V(s_{\pi\pi})$ is known very well from both direct measurements of $e^+e^- \rightarrow \pi^+\pi^-$ [10,11,12,13] as well as theoretical studies [14,15,16,17,18,19,20]. It collects all non-perturbative $\pi\pi$ interactions and is universal. On the other hand, the function $P(s_{\pi\pi})$ as well as the normalization factor A are reaction specific and – at least for the decay of light mesons or, more accurately, for small values of $s_{\pi\pi}$ – are expected to be perturbative in the sense of ChPT. In case of the η and η' decays in the focus here, left-hand cut contributions are suppressed both kinematically, since the particle pairs in the t -channel are to be (at least) in a p -wave, and chirally, since the p -wave $\pi\eta$ interaction starts at next-to-leading order only [21,22]. We may therefore expand $P(s_{\pi\pi})$ in a Taylor series around $s_{\pi\pi} = 0$ and define

$$P(s_{\pi\pi}) = 1 + \alpha s_{\pi\pi} + \mathcal{O}(s_{\pi\pi}^2) . \quad (2)$$

At higher order non-analytic terms, mainly from left-hand cuts, need to be considered. The parameters α and A allow insights into the physics underlying the decay process. Thus, in the *second step* of our method a proper matching scheme is formulated, to relate the parameters A and α to the parameters of the underlying effective field theory. For the example at hand we will find that this matching allows us, under certain assumptions, to impose a relation between $\eta \rightarrow \pi^+\pi^-\gamma$ and $\eta' \rightarrow \pi^+\pi^-\gamma$.

The paper is structured as follows: after a brief discussion of the pion vector form factor, we apply the formalism to the mentioned η and η' decays in Sec. 3, extracting the phenomenological parameters $\alpha^{(\prime)}$ and $A^{(\prime)}$. In Sec. 4 those are then interpreted via a matching to one-loop $U(3)$ extended ChPT. In Sec. 5 an interpretation of essentially half of the empirical value of $\alpha^{(\prime)}$ is given, whereas Sec. 6 contains a comparison to earlier studies. We close with a summary.

2. We start with a brief discussion of the pion vector form factor. In terms of the vector-isovector current V_μ^3 , it is defined via

$$\langle \pi^+(p')\pi^-(p)|V_\mu^3|0\rangle = (p-p')_\mu F_V(s_{\pi\pi}) . \quad (3)$$

In the elastic regime the form factor is, in symbolic notation, defined via

$$F_V = M_V + T_{\pi\pi} G_{\pi\pi} M_V , \quad (4)$$

with M_V , $G_{\pi\pi}$ and $T_{\pi\pi}$ for the production vertex, the two-pion propagator and the $\pi\pi$ scattering amplitude, respectively. For simplicity, in this work we assume M_V to be real which is exact for the transitions we will study explicitly

below. Vector meson dominance models typically model the form factor, either, in its simplest variant by a single term, $F_V(s_{\pi\pi}) = -m_\rho^2/(s_{\pi\pi} - m_\rho^2 + im_\rho\Gamma_\rho)$ where m_ρ and Γ_ρ are the ρ mass and width, respectively, or by writing the second term of Eq. (4) as a (sum of) vector meson propagator(s) times $s_{\pi\pi}$, which leaves the relative strength of the first and the second term as free parameter. We will here take a different route which leaves no freedom of choice. From the definition of Eq. (4) it follows straight forwardly that

$$\text{Im}(F_V(s_{\pi\pi})) = \sigma(s_{\pi\pi}) T_{\pi\pi}(s_{\pi\pi})^* F_V(s_{\pi\pi}) . \quad (5)$$

This relation holds for the whole elastic regime which, in case of the pion vector form factor, extends to values of $s_{\pi\pi}$ well beyond 1 GeV^2 , although already at $s_{\pi\pi} = 16m_\pi^2$ formally the first inelasticity opens. Eq. (5) is one way to present the Watson theorem [23]: since $\text{Im}(F_V)$ is a real quantity, the phase of the form factor has to agree with the phase of the elastic scattering amplitude. In the elastic regime this equation can be written as $\text{Im}(F) = \tan \delta_{11} \text{Re}(F)$, with δ_{11} for the elastic $\pi\pi$ phase shift in the vector channel. The resulting dispersion integral can be solved analytically to give

$$F_V(s_{\pi\pi}) = \exp \left(\frac{1}{6} s_{\pi\pi} \langle r^2 \rangle + \frac{s_{\pi\pi}^2}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\delta_{11}(s)}{s^2(s - s_{\pi\pi} - i\epsilon)} \right) . \quad (6)$$

Contrary to the standard procedure [16], we follow Refs. [19,20] and use a twice subtracted dispersion integral in order to guarantee that the integral over the phase converges in the elastic regime. The phase δ_{11} can be taken from data or from theoretical analysis. The phase used in the present analysis is taken from Ref. [17] (see Eqs. (A7) and (A8) therein), valid up to $\sqrt{s_{\text{cut}}} = 1.4 \text{ GeV}$ — we use a smooth extrapolation of the phases beyond this energy following Ref. [18]. The phases agree with those of Ref. [24] up to 800 MeV and with the available data. Once the phase is fixed, Eq. (6) has only one free parameter — the subtraction constant $\langle r^2 \rangle$, the mean square charge radius of the pion. Apart from the region around $s_{\pi\pi} = m_\omega^2$, where $\rho - \omega$ mixing shows up very prominently, we find an excellent fit of the data for $\langle r^2 \rangle = 0.437(3) \text{ fm}^2$. The uncertainty includes the uncertainty in the $\pi\pi$ phase shifts given in Ref. [17] as well as the weak dependence on s_{cut} . The form factor parametrization, including the uncertainty, is shown as the red band in Fig. 1. The range for the radius is consistent with $\langle r^2 \rangle = 0.452(13) \text{ fm}^2$ of Ref. [25] and $\langle r^2 \rangle = 0.435(2) \text{ fm}^2$ of Ref. [18]. The possible impact of $\rho - \omega$ mixing on the η' decay spectra is briefly discussed below.

As it provides the explicit solution for F_V defined in Eq. (4), Eq. (6) contain both the Born term (pions going out without interaction) as well as the final state interaction. In general, the mentioned dispersion integral fixes the form factor up to a multiplicative function that does not have right-hand cuts. Since the right-hand discontinuity of the transition amplitude $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$

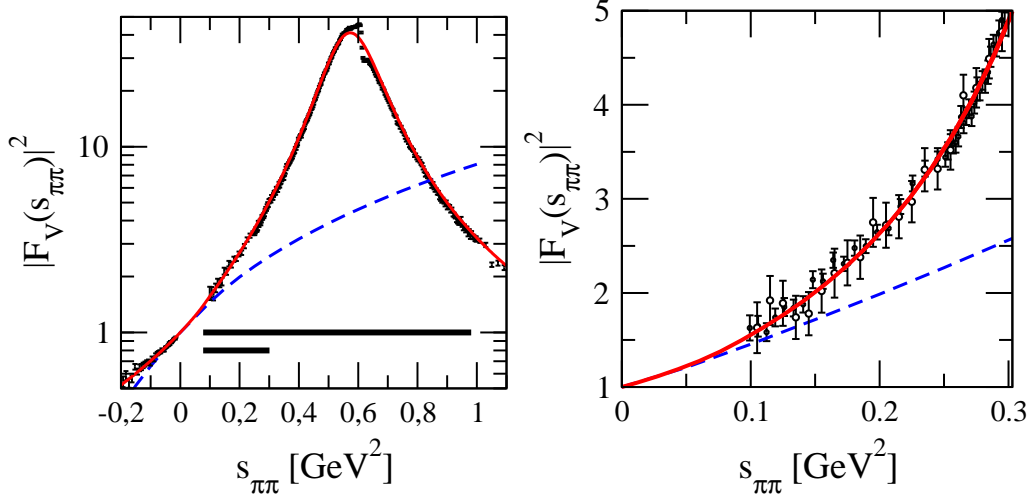


Fig. 1. The (red) solid band shows the form factor derived from Eq. (6), the (blue) dashed line the result from one-loop ChPT (see Eq. (7))— both with identical values for the pion radius. The time-like data from Refs. [11] and [12] are shown as solid and open circles, respectively. The space-like data are from Ref. [13]. The short (long) thick, horizontal bar in the left panel denotes the kinematic range covered in the decay of the η (η'). The right panel shows, as a linear plot, a zoom into the $s_{\pi\pi}$ range relevant for the η decay. In this energy range the form factor can be approximated by the polynomial $|F_V(s_{\pi\pi})| \approx 1 + (2.12 \pm 0.01)s_{\pi\pi} + (2.13 \pm 0.01)s_{\pi\pi}^2 + (13.80 \pm 0.14)s_{\pi\pi}^3$ with $s_{\pi\pi}$ measured in units of GeV^2 .

agrees with that of the pion vector form factor, the factorization employed in Eq. (1) is justified — under the above-stated qualifications that the higher partial waves can be neglected and that the left-hand cut contributions are suppressed. We chose the standard normalization for the form factor, namely $F_V(0)=1$, which corresponds to $M_V = 1$ in Eq. (4).

Remember that to one-loop order the expression for the form factor reads [26]

$$F_V(s_{\pi\pi}) = 1 + \frac{1}{6f_\pi^2}(s_{\pi\pi} - 4m_\pi^2)\bar{J}(s_{\pi\pi}) + \frac{s_{\pi\pi}}{6} \left(\langle r^2 \rangle + \frac{1}{24\pi^2 f_\pi^2} \right), \quad (7)$$

where the function \bar{J} is defined in [26] and where $f_\pi = 92.2$ MeV denotes the pion decay constant. The small kaon loop contribution is taken care of by the use of the empirical value of the pion squared radius.

The full expression for the form factor, Eq. (6), and its one-loop counterpart, Eq. (7), which enters the 1-loop ChPT prediction for $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$ of Ref. [3]¹ are compared with data in Fig. 1. While by construction both curves correspond to the same pion radius, the two descriptions start to deviate already

¹ Actually, Ref. [3] discusses the radiative two-pion decays of the octet and singlet states, η_8 and η_1 .

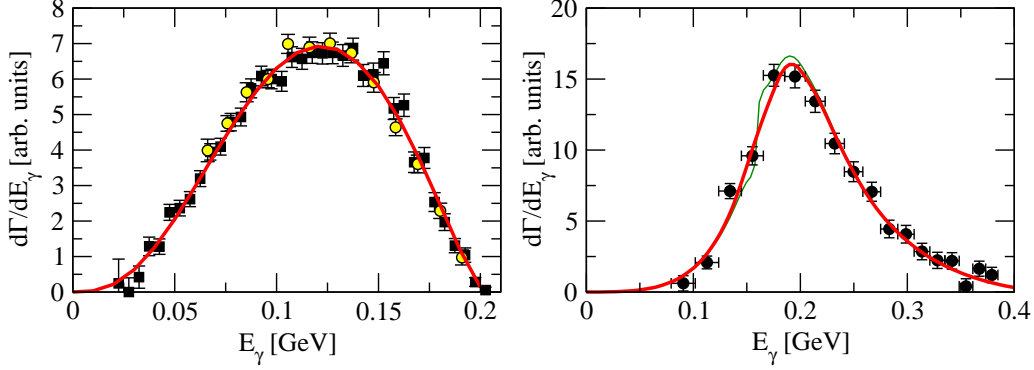


Fig. 2. Experimental data and error weighted fits for η (left, data are from Ref. [27] (filled squares) and Ref. [28] (open circles)) and η' (right, data are from Ref. [29]) to $\pi^+\pi^-\gamma$ according to Eqs. (1) and (2) with $s_{\pi\pi} = m_{\eta^{(\prime)}}(m_{\eta^{(\prime)}} - 2E_\gamma)$. The thin (green) line in the right panel denotes the possible impact of $\rho - \omega$ mixing under the assumption that it appears here with the same strength as in F_V .

visibly at values of $s_{\pi\pi}$ as low as 0.09 GeV^2 . The kinematic range covered in the radiative η and η' decays spans from 0.077 GeV^2 to 0.301 GeV^2 and 0.918 GeV^2 , respectively. The two ranges are indicated in the figure by the thick bars. While one might hope to be able to describe the form factor in the kinematic range for the η within ChPT at sufficiently high orders (*e.g.* below 0.25 GeV^2 the 2-loop order seems to be sufficient [14,15]), clearly a description for the η' is impossible with any perturbative series.

As we will demonstrate below: once the non-perturbative part of the transition amplitude in form of the pion vector form factor is divided out, what remains can be described by a polynomial in $s_{\pi\pi}$ of low order that can be analyzed within $U(3)$ extended ChPT, even in the case of the radiative η' decay.

3. We now turn to the evaluation of the full decay amplitudes. As mentioned above the first step is to analyze the data for both the total decay rates as well as the spectra with a fit of the parameters A and α of Eq. (1). Note that the experiments discussed below confirmed that the pion pair is in the vector isovector channel. The resulting fits to the spectra of the WASA-at-COSY collaboration [27] for the η case and of the CRYSTAL BARREL collaboration [29] for the η' case are shown in Fig. 2, left and right panel, respectively. From the error weighted fits, we extract values of

$$\alpha = (1.96 \pm 0.27 \pm 0.02) \text{ GeV}^{-2} ; \quad \alpha' = (1.80 \pm 0.49 \pm 0.04) \text{ GeV}^{-2} \quad (8)$$

where the parameter extracted from the data on the η' appears as primed. The first and second uncertainty originate from the fit to the data on $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$ and from that of the pion vector form factor, respectively. The α parameter was also determined directly by the WASA-at-COSY collaboration quoting in addition a systematic uncertainty of 0.59 GeV^{-2} . Since the CRYSTAL BARREL data points include systematic uncertainties, the uncertainty of the α'

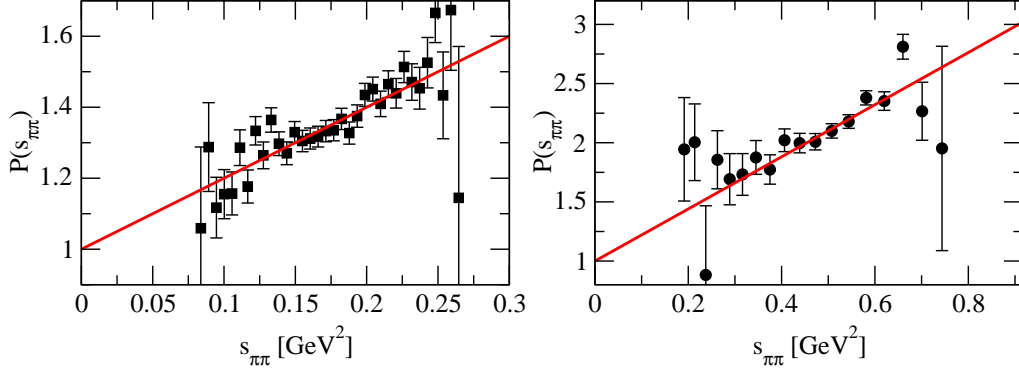


Fig. 3. Plot of $P(s_{\pi\pi})$, defined in Eqs.(1) and (2), as extracted from data for $\eta \rightarrow \pi^+\pi^-\gamma$ [27] (left panel) and $\eta' \rightarrow \pi^+\pi^-\gamma$ [29] (right panel).

value should include both statistical and systematic uncertainties. We also studied other data sets for η and η' . Concerning the former decay, Gormley *et al.* [28] provides $\alpha = (1.8 \pm 0.4) \text{ GeV}^{-2}$ while Layter *et al.* [30] gives $\alpha = (-0.9 \pm 0.1) \text{ GeV}^{-2}$. The acceptance correction of these old experiments was derived from the specified $d\Gamma/dE_\gamma$ distributions, respectively, under the assumption that the pertinent matrix element is the simplest gauge invariant one (corresponding here to $P(s_{\pi\pi})$ and $F_V(s_{\pi\pi})$ equal to one). The Layter *et al.* result seems to be inconsistent both with WASA [27] and Gormley *et al.* [28]. However, from the information provided in those old experimental papers it is impossible to evaluate systematic uncertainties. In case of the η' , we obtain $\alpha' = (2.7 \pm 1.0) \text{ GeV}^{-2}$ from the data of the GAMS-200 collaboration [31], which is larger, but within error bars consistent with the value listed above. Hence, in the following, we use the values given in Eq. (8).

In the pion vector form factor $\rho - \omega$ mixing shows up as a quite spectacular effect. In case of the η' decay, as a result of the E_γ^3 behavior of the rate at low values of E_γ , the effect is a lot smaller: if we take the mixing effect with the same strength as it appears in F_V using the prescription of Ref. [18], the impact of the mixing on the η' spectrum is very moderate — see the wiggly (green) line in the left panel of Fig. 2. A fit to the η' spectrum including the mixing as shown shifts the value of α' upwards by 0.3 GeV^{-2} , well within errors.

Instead of looking at the data themselves it is illustrative to extract from data directly the polynomials $P(s_{\pi\pi})$. These are shown for both radiative η and η' decays in the left and right panel of Fig. 3, respectively. Here one clearly sees that the residual $s_{\pi\pi}$ dependence for both transition amplitudes — once the pion form factor and the phase space are divided out — has a linear behavior to a very good approximation. The statement is further corroborated by the fact that any *additional* quadratic term to the linear polynomial with coefficients as specified in Eq. (8) is compatible with zero: $\beta = (0.21 \pm 0.67 \pm 0.06) \text{ GeV}^{-4}$ and $\beta' = (0.04 \pm 0.36 \pm 0.03) \text{ GeV}^{-4}$. This appears reassuring, although it

came as a surprise that even for the η' a first-order polynomial is sufficient. The origin of this might be in the current quality of the data which is best in the region of large values of E_γ which corresponds to moderate values of $s_{\pi\pi}$ — this is the region where the chiral expansion is expected to converge (once resonance effects are taken out). This can also be seen in Fig. 3, right panel: clearly the fit is dominated by values of $s_{\pi\pi} \leq 0.6 \text{ GeV}^2$ (this corresponds to pion relative momenta of at most 360 MeV) that are still reasonably smaller than the typical hadronic scale of order of 1 GeV, which sets the expected range of convergence for the chiral expansion once the vector pion form factor is taken out.

The normalization of the data used above cannot be employed to fix the prefactor $A^{(\prime)}$, for they are given in arbitrary units only. However, once the slope parameters are fixed we can use the experimentally measured partial widths [32], $\Gamma_{\eta \rightarrow \pi\pi\gamma}^{exp.} = (59.8 \pm 3.8) \text{ eV}$ and $\Gamma_{\eta' \rightarrow \pi\pi\gamma}^{exp.} = (57.0 \pm 2.8) \text{ keV}$, to extract

$$\delta = -0.22 \pm 0.04 ; \quad \delta' = -0.40 \pm 0.09 , \quad (9)$$

where we used the definition $A^{(\prime)} = A_0^{(\prime)}(1 + \delta^{(\prime)})$, with $A_0^{(\prime)}$ for the transition strength in the chiral limit², which is fixed for the transitions at hand by the chiral (box) anomaly [3,7].

It should be stressed that the values extracted for $\alpha^{(\prime)}$ are a lot more reliable at this stage than those extracted for $\delta^{(\prime)}$, since the former are differential quantities while the latter are integrated ones, being quite sensitive to the shape of the spectrum also in the regime where we do not have high quality data yet. Thus, one might expect that, once these become available also for the η' decay spectra at large values of $s_{\pi\pi}$, a linear polynomial is insufficient for $P(s_{\pi\pi})$. This should not change $\alpha^{(\prime)}$, however, it might significantly influence the integrated rate and therefore the values of $\delta^{(\prime)}$.

4. In order to perform the matching of the decay amplitude to one-loop ChPT we now replace F_V in Eq. (1) by its one-loop expression, Eq. (7), and expand the result to first order in $s_{\pi\pi}$. The resulting expression can then be equated with the corresponding one from one-loop ChPT, see appendix A. This procedure gives a relation between the phenomenological parameters $\alpha^{(\prime)}$ and $\delta^{(\prime)}$ and the $U(3)$ ChPT low-energy parameters $a_1^{(8\pi)}$, $a_1^{(8K)}$, $a_1^{(1\pi)}$, $a_1^{(1K)}$ and a_2 .

² Here and in the following we will use the term ‘chiral limit’ in a somewhat loose sense, since we insert in A_0 the physical value of the pion decay constant f_π and assume that $f_8 \neq f_\pi$ for the octet pseudoscalar decay constant, etc.

Especially one finds from the $s_{\pi\pi}$ independent terms for the η -decay

$$\delta = \frac{1}{32\pi^2 f_\pi^2} \left[\frac{A_8}{A_0} \left(a_1^{(8\pi)} m_\pi^2 + a_1^{(8K)} m_K^2 \right) + \frac{A_1}{A_0} \left(a_1^{(1\pi)} m_\pi^2 + a_1^{(1K)} m_K^2 \right) - 4m_\pi^2 \log \frac{m_\pi^2}{\mu^2} + \frac{4A_8 - 2A_1}{A_0} m_K^2 \log \frac{m_K^2}{\mu^2} \right] \quad (10)$$

and from the $s_{\pi\pi}$ dependent terms for the η -decay

$$\alpha + \frac{1}{6} \langle r^2 \rangle = \frac{1}{32\pi^2 f_\pi^2} \left[a_2 - \frac{1}{3} \log \frac{m_\pi^2}{\mu^2} - \frac{4A_8 + A_1}{6A_0} \log \frac{m_K^2}{\mu^2} - \frac{1}{9} \left(8 + \frac{2A_8 - 7A_1}{2A_0} \right) \right]. \quad (11)$$

Here m_K denotes the kaon mass. In line with previous investigations we identify the scale μ with m_ρ . Furthermore we introduced the abbreviations

$$A_8 = a \frac{e}{4\sqrt{3}\pi^2 f_\pi^3 f_8}; \quad A_1 = b \frac{e\sqrt{2}}{4\sqrt{3}\pi^2 f_\pi^3 f_1}$$

with e the unit of electric charge, $a = \cos(\theta)$ and $b = -\sin(\theta)$ in terms of $\theta \simeq -20^\circ$ as the value of the $\eta - \eta'$ mixing angle and f_8 and f_1 the values of the octet and singlet pseudoscalar decay constants – all three values as specified as in Refs. [6,7], see also Ref. [3]. Since our aim here is the comparison with the existing radiative decay results of the above mentioned references, we still follow the old parameterization in terms of two (octet and singlet) decay constants and *one* $\eta - \eta'$ mixing angle. This is in contrast to more modern parameterizations in terms of either two octet-singlet decay constants and two mixing angles, which follow from the defining matrix elements of the octet and singlet axial-vector current [33], or in terms of strange and non-strange decay constants and only one mixing angle, see *e.g.* Refs. [34,35,36].

The transition strength in the chiral limit is given by $A_0 = A_8 + A_1$. To arrive at the corresponding expressions for the η' decays, α , δ , A_0 , a and b need to be replaced by their primed counter parts, especially $a' = \sin(\theta)$ and $b' = \cos(\theta)$. In Eq. (11) we have only kept the leading term of the kaon loop contributions, which turns out to represent the complete expression to high accuracy.

In the case of the $s_{\pi\pi}$ independent terms, we observe that two parameters from the phenomenological expression, δ and δ' , have to be matched onto four parameters from $U(3)$ extended ChPT, $a_1^{(8\pi)}$, $a_1^{(8K)}$, $a_1^{(1\pi)}$ and $a_1^{(1K)}$. Thus the matching does not provide any constraint. The situation is different for the $s_{\pi\pi}$ dependent terms, as the two parameters α and α' have to be matched to the single parameter a_2 . To justify this statement we needed to ignore $s_{\pi\pi}$ dependent counter terms subleading in N_c , the number of colors. Note, however, that these neglected terms are used to cancel some of the additional

divergences in the one-loop approximation of the $\eta_1 \rightarrow \pi^+\pi^-\gamma$ decay [3]. Extracting a_2 from the η and the η' decays gives

$$a_2^\eta = 9.70 \pm 0.7 ; \quad a_2^{\eta'} = 9.23 \pm 1.4 ,$$

respectively. Thus we indeed find that the data sets for the η and η' decays are consistent with the assumption that the only non-perturbative part of the amplitude originates from the pion vector form factor.

5. In the following, by invoking chiral Ward identity and large N_c arguments, we will give a physical interpretation to essentially half of the parameter $\alpha^{(\prime)}$. When the hidden gauge approach [37,38] to low-energy hadron physics was first applied to the anomalous sector, it was already found out that the so-called *complete* VMD (for both the triangle and the box anomaly sector) was unsustainable in this approach [39], see also the reviews [40,41,42]: while the non-anomalous and the triangle-anomaly sectors were fully compatible with VMD, the description of the box-anomaly-induced decays gave satisfying predictions (see *e.g.* the $\omega \rightarrow \pi\pi\pi$ decay) only if the VMD triangle anomaly terms were supported by a point-vertex (contact term) involving a photon and three pseudoscalars. Cohen [43] was the first (see also [44]) to point out that the chiral Ward identity implies that both the chiral triangle and the box anomaly contribute to the $\gamma\pi\pi\pi$ processes and the $\eta^{(\prime)} \rightarrow \pi\pi\gamma$ decays away from the chiral limit. With the help of the low-energy theorem for the $\gamma \rightarrow \pi\pi\pi$ amplitude of Ref. [45], he argued that the total amplitude for these (in the chiral limit by the box-anomaly induced) processes can be decomposed as

$$\mathcal{A}^{\text{tot}} = \frac{3}{2}\mathcal{A}^{\text{VVA}} - \frac{1}{2}\mathcal{A}^{\text{VAAA}} , \quad (12)$$

where the triangle or VVA-type amplitude [V: vector, A: axialvector], under the assumption that *one* of the vectors of the VVA-type amplitude subsequently decays into two pions, and the box or VAAA-type amplitude contribute with a relative weight of $\frac{3}{2} : -\frac{1}{2}$. Still there is an ongoing debate whether these VAAA-type contact terms are empirically needed or not, see *e.g.* [46,5].

The vector pion form factor $F_V(s_{\pi\pi})$, which – as mentioned before – contains both the Born terms and final state $\pi\pi$ interactions, can be represented by one of the V legs in the triangle anomaly VVA vertex (the other V leg denotes the outgoing (isovector) photon, whereas the A leg stands for the decaying pseudoscalar η under PCAC). In other words the VVA vertex

$$\frac{3}{2}\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)F_V(s_{\pi\pi})\tilde{P}(s_{\pi\pi}) \quad (13)$$

(with $\tilde{P}(s_{\pi\pi}) = 1 + \tilde{\alpha}s_{\pi\pi} + \mathcal{O}(s_{\pi\pi}^2)$ a polynomial, similar to $P(s_{\pi\pi})$ in Eq. (1)) contains both *tree-level* contributions of the vector form factor, which are of leading order in the $1/N_c$ expansion, and *genuine loop* contributions of F_V , which are suppressed by at least one factor of $1/N_c$ and are therefore

subleading³. In the case of the VAAA box anomaly, however, the two remaining AA legs stand for two *separate* pions which either are already *the final pions* or which still *rescatter*. The first case, which is the leading- N_c contribution of the VAAA process (times the $\mathcal{O}(N_c^{-1/2})$ scaling of the prefactor $\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)$), would correspond in the VVA scenario just to the replacement $F_V(s_{\pi\pi}) \rightarrow 1$, *i.e.* to the trivial term in the Taylor-expansion of the vector form factor (6), while the second (rescattering) case, since it necessarily involves an additional four-pion vertex, is represented by the *subleading* terms in the $1/N_c$ expansion of $F_V(s_{\pi\pi})$. The leading-order terms of the VVA- and VAAA-type amplitudes have to have the same N_c scaling, since both are *tree-level* processes and since the initial state ($\eta^{(\prime)}$) and the final state ($\pi^+\pi^-\gamma$) are the same, respectively. However, only the former can run through a vector meson pole at tree-level. The total VAAA result is therefore

$$-\frac{1}{2}\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)e^{-\frac{1}{6}\widetilde{\langle r^2 \rangle}s_{\pi\pi}}F_V(s_{\pi\pi})\tilde{P}(s_{\pi\pi}) . \quad (14)$$

Here the coefficient $\widetilde{\langle r^2 \rangle}$ is the leading N_c^0 contribution of the mean square charge radius of the pion, which to $\mathcal{O}(m_\pi^2)$ in ChPT is given by $\langle r^2 \rangle = (\bar{l}_6 - 1)/(16\pi^2 f_\pi^2)$, see *e.g.* Ref. [20], such that $\widetilde{\langle r^2 \rangle} = (\bar{l}_6 - \bar{l}_6^{N_c^0})/(16\pi^2 f_\pi^2)$. Remember that \bar{l}_6 itself is of order N_c and that f_π scales as $\sqrt{N_c}$. The exponential term $\exp(-\frac{1}{6}\widetilde{\langle r^2 \rangle}s_{\pi\pi})$ in the VAAA vertex therefore serves to remove the leading N_c^0 contributions in $F_V(s_{\pi\pi})$, such that – with exception of the trivial term – only the subleading ones are left over (multiplying $-\frac{1}{2}\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)$). Summing the VAA and VAAA induced contributions to the $\eta^{(\prime)} \rightarrow \pi\pi\gamma$ decay amplitude, *i.e.* (13) and (14), and Taylor-expanding the exponential factor of (14) and $\tilde{P}(s_{\pi\pi})$, we therefore get the total amplitude

$$\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(s_{\pi\pi}) = \mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)F_V(s_{\pi\pi}) \left[1 + \underbrace{\left(\frac{1}{12}\widetilde{\langle r^2 \rangle} + \tilde{\alpha} \right)}_{\equiv \alpha} s_{\pi\pi} + \mathcal{O}(s_{\pi\pi}^2) \right] . \quad (15)$$

Thus, from this point of view, the coefficient α comprises of two terms: the leading N_c^0 term of the subtraction constant of the vector form factor, $\frac{1}{2}\widetilde{\langle r^2 \rangle}/6 \approx \bar{l}_6/(192\pi^2 f_\pi^2) \approx 1.03 \text{ GeV}^{-2}$ [20], and the remainder $\tilde{\alpha}$. Comparing to the numbers of Eq. (8), we find in this case that about half of the value of $\alpha^{(\prime)}$ can be interpreted as coming from $\widetilde{\langle r^2 \rangle}$.

6. Finally, we will link our method to earlier studies and compare results with vector meson dominance considerations in general. Inspired by an N/D analysis of the process $\gamma \rightarrow \pi\pi\pi$ (see also [8,46]), Holstein and Venugopal [6,7]

³ Note that the prefactor $\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)$ contains the coefficients $N_c/f_\pi^3 \sim \mathcal{O}(N_c^{-1/2})$ and therefore determines the leading overall N_c scaling behavior of the total $\eta^{(\prime)} \rightarrow \pi\pi\gamma$ decay amplitude. It is of course the same for the VVA and the VAAA case.

constructed a precursor to the relation (1). They started from an ansatz containing a contact term as well as a rescattering term parameterizing the $\pi\pi$ final state interactions, which reads in the notation used above

$$\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(s_{\pi\pi}) = \mathcal{A}_{\eta \rightarrow \pi\pi\gamma}^0 [1 - c + c(1 + as_{\pi\pi})F_V(s_{\pi\pi})] . \quad (16)$$

Here c and a are free real-valued parameters and $\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}^0$ is the amplitude in the chiral limit. By matching this ansatz – both at order $\mathcal{O}(p^6)$ – to one-loop chiral perturbation theory [3] (more precisely, to the coefficient in Ref. [3] of the standard one-loop function $\bar{J}(s_{\pi\pi})$, cf. Eq. (7)) on the one hand and to VMD (as in Ref. [4]) on the other hand, these free parameters were fixed in Refs. [6,7] to $c = 1$ via ChPT and to $a = 1/(2m_\rho^2)$ via VMD, respectively. In light of the discussion presented above and as implicitly stated in Refs. [6,7], $c = 1$ (*i.e.* no contact term) appears as a necessity from dispersion theory. Note that Ko and Truong [8] derived the same expression from unitarity and the above-discussed Ward identity constraints. If the parameter a in Refs. [6,7] were not extracted from VMD or $\alpha' = 0$ were not implicitly assumed as in Ref. [8] and the amplitude $\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}(0)$ were not fixed to its chiral limit value $\mathcal{A}_{\eta \rightarrow \pi\pi\gamma}^0$, then the resulting relations of Refs. [6,7] and of Ref. [8] would have had the form of Eq. (1), for $P(s_{\pi\pi})$ expanded to first order.

Let us now compare to vector meson dominance in general: in its most simple form one would just get $\alpha = 0$, at variance with data, see (8). If, following Ref. [43], the chiral Ward identities are implemented, then the simplest scenario would correspond to $\tilde{\alpha} = 0$ instead of $\alpha = 0$, a result at the edge of being consistent with current data. This is what was used in Refs. [6,7,8]. Finally note that in all evaluations that modify the $\eta \rightarrow \pi^+\pi^-\gamma$ decay amplitude of the chiral limit solely by $s_{\pi\pi}$ dependent terms, as is the case in vector meson dominance models, it is impossible to simultaneously predict the experimentally measured shape of the distribution and the empirical branching ratio.

7. To summarize, the $\eta^{(\prime)} \rightarrow \pi^+\pi^-\gamma$ decay amplitude, here assumed to be p -wave dominated, factorizes into a universal part and a reaction specific part after the trivial (point-particle) kinematics is removed. If higher partial waves cannot be neglected, their contribution should be added suitably to the above-mentioned p -wave amplitude. The universal part, which applies to all p -wave dominated radiative decays with an isovector $\pi^+\pi^-$ pion pair in the final state, is given by the well established vector pion form factor F_V . The reaction specific part can be parametrized as an expansion in the invariant mass $s_{\pi\pi}$ of the pion pair: $P(s_{\pi\pi})$ times the normalization factor A — as long as left-hand cut contributions are suppressed, as is the case for the reactions under consideration. The expansion can then be treated perturbatively, since unitarity and analyticity of the final-state interaction dictate that the pion form factor already takes care of the non-perturbative aspects of the $\pi\pi$ unitary cut.

Moreover, the perturbative expansion allows a systematic comparison with ChPT predictions. Especially, the case with η' decays shows a way to extend methodically perturbative calculations also to the resonance region. In this sense the presented approach is indeed *model-independent*.

For the description of the present world data it is sufficient to consider only the linear term (the $\alpha^{(\prime)}$ parameter) in the expansion of $P(s_{\pi\pi}) = 1 + \alpha^{(\prime)} s_{\pi\pi} + \mathcal{O}(s_{\pi\pi}^2)$ — both for the η and η' decays. The extracted values of this parameter for the existing experiments are $\alpha = (1.96 \pm 0.27_{\text{stat}} \pm 1.00_{\text{syst}}) \text{ GeV}^{-2}$ and $\alpha' = (1.80 \pm 0.49_{\text{tot}}) \text{ GeV}^{-2}$. The data of Gormley *et al.* [28] and the new data of the WASA-at-COSY collaboration [27] are consistent (both support an α parameter of the order 2 GeV^{-2}), whereas the data of Layter *et al.* [30] indicate an α value below zero. However, the old experiments (Gormley and Layter) do not provide estimates of the systematic uncertainties.

We propose to use the introduced α and α' coefficients or, more generally, the polynomial $P(s_{\pi\pi})$ to parametrize and compare – including the pertinent statistical and systematical uncertainties – the spectra of all past and future radiative decay experiments with only one (iso)vector p-wave $\pi^+\pi^-$ pair in the final state. New experimental data on the $\eta \rightarrow \pi^+\pi^-\gamma$ decay distributions from KLOE are at the final stages of the analysis [47] and will be released soon. Meanwhile also WASA-at-COSY, CLAS and BES-III have collected new large data sets of respectively η and η' decays that will enable us to check if further terms in the expansion of $P(s_{\pi\pi})$ are necessary.

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A Explicit expressions from one-loop $U(3)$ extended ChPT

The ChPT amplitude employed in the matching (*cf.* Eqs. (10, 11)) has been obtained as follows. The divergences occurring in the regularized one-loop expression [3] are absorbed by anomalous $\mathcal{O}(p^6)$ counter terms – a complete

list of which is presented in [48,49] in the $SU(3)$ case. We follow Ref. [9] where also some $U(3)$ extensions of these terms can be found. However, we neglect the pure singlet terms containing the coefficients \bar{W}_{13} and \bar{W}_{14} of Ref. [9], which posses an $s_{\pi\pi}$ -dependence, but have one more trace than their $SU(3)$ counterparts, such that they are only subleading in the $1/N_c$ expansion. Furthermore, we use the Gell-Mann–Okubo formulae to dispose the η [50] and η' [51] masses, which is correct to $\mathcal{O}(p^6)$. In summary, we get the ChPT amplitude

$$\mathcal{A}^{\text{ChPT}} = [A_8 \cdot C_{\eta_8} + A_1 \cdot C_{\eta_1}] \epsilon_{\mu\nu\alpha\beta} (\epsilon_\gamma^*)^\mu (p_\gamma)^\nu (p_+)^{\alpha} (p_-)^{\beta} \quad (\text{A.1})$$

with

$$C_{\eta_8} = 1 + C_{\eta_8}^{\text{loops}} + \frac{1}{32\pi^2 f_\pi^2} [a_1^{(8\pi)} m_\pi^2 + a_1^{(8K)} m_K^2 + a_2 s_{\pi\pi}] , \quad (\text{A.2})$$

$$C_{\eta_1} = 1 + C_{\eta_1}^{\text{loops}} + \frac{1}{32\pi^2 f_\pi^2} [a_1^{(1\pi)} m_\pi^2 + a_1^{(1K)} m_K^2 + a_2 s_{\pi\pi}] . \quad (\text{A.3})$$

Here the coefficients $C_{\eta_i}^{\text{loops}}$ comprehend the finite loop contributions (see [3]), while the remaining constants comprise the low-energy constants (LECs) of Ref. [9] in the following way:

$$a_1^{(8\pi)} = \kappa \cdot \left[-4\bar{w}_7^{(0)} - 8\bar{w}_8^{(0)} - \frac{7}{3}\bar{w}_{11}^{(0)} - 2\bar{w}_{12}^{(0)} \right] , \quad (\text{A.4})$$

$$a_1^{(8K)} = \kappa \cdot \left[8\bar{w}_8^{(0)} + \frac{4}{3}\bar{w}_{11}^{(0)} \right] , \quad (\text{A.5})$$

$$a_1^{(1\pi)} = \kappa \cdot \left[-4\bar{w}_7^{(0)} + 2\bar{w}_8^{(0)} + 12\bar{w}_9^{(0)} + 3\bar{w}_{10}^{(0)} - 5\bar{w}_{11}^{(0)} - 2\bar{w}_{12}^{(0)} \right] , \quad (\text{A.6})$$

$$a_1^{(1K)} = \kappa \cdot \left[4\bar{w}_8^{(0)} + 6\bar{w}_{10}^{(0)} + 4\bar{w}_{11}^{(0)} \right] , \quad (\text{A.7})$$

$$a_2 = \kappa \cdot \left[2\bar{w}_{11}^{(0)} + \bar{w}_{12}^{(0)} \right] \quad (\text{A.8})$$

with $\kappa = 2^{11}\pi^4 f_\pi^2$. Hence the momentum dependent terms of η_8 and η_1 involve only *one* linear combination of the LECs (*cf.* a_2), whereas the mass terms are governed by *four* linear combinations.

Note that the structure of C_{η_8} and C_{η_1} displayed in Eqs. (A.2) and (A.3) can also be derived from [48,49], if the Gell-Mann–Okubo formulae and the pertinent $U(3)$ extension are applied as above.

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